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ELECTROMAGNETIC RADIATION FROM A VIBRATING QUARTZ PLATE

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Abstract—The linear equations of piezoelectromagnetism are solved for the case of thickness-shear vibrations of a quartz plate. For an AT-cut plate vibrating near resonance, the radiation from each face is about 25 μW/cm^2 for a shear strain of 10^{-5}.

1. INTRODUCTION

THE problem of electromagnetic radiation from a quartz plate vibrating in a predominantly thickness-shear mode was studied by Tiersten [1] who concluded that there is no radiation in the case of a "trapped energy" [2] mode. In the present paper, an exact solution of the linear equations of piezoelectromagnetism is obtained for a mode of vibration favorable to radiation: thickness-shear, independent of the coordinates in the plane of the plate. A simple formula is obtained for the power radiated, per unit of area, from each face of the plate:

$$P = \tfrac{1}{2}\varkappa_{33}\hat{c}_{66}^2 S_0^2/\rho c,$$

where $\varkappa_{33}$ is a dielectric constant, $\hat{c}_{66}$ is a shear stiffness, S_0 is the maximum shear strain, ρ is the mass density and c is the velocity of light *in vacuo*. For the AT-cut of quartz, the radiation is about 25μW/cm^2 for a shear strain of 10^{-5}.

2. EQUATIONS AND BOUNDARY CONDITIONS

The field equations of piezoelectromagnetism in a dielectric comprise the stress equations of motion

$$T_{ij,i} = \rho \ddot{u}_j \tag{1}$$

and the equations of the electromagnetic field:

$$\varepsilon_{ijk}E_{k,j} = -\dot{B}_i, \qquad \varepsilon_{ijk}B_{k,j} = \mu_0\dot{D}_i, \qquad B_{i,i} = 0, \qquad D_{i,i} = 0. \tag{2}$$

In these equations, $\mathbf{u}$ is the mechanical displacement, $\mathbf{E}$ is the Maxwell electric self-field, $\mathbf{B}$ is the magnetic flux density, ρ is the mass density, μ_0 is the magnetic permeability of a vacuum and ε_{ijk} is the unit alternating tensor. For the rotated-Y-cuts of quartz [3], the components of stress, T_{ij}, and the components of electric displacement, D_i, are expressed, in the linear theory, in terms of the components of strain,

$$S_{ij} = \tfrac{1}{2}(u_{j,i} + u_{i,j}), \tag{3}$$

and the components of the electric field, E_i, according to

$$\begin{aligned}
T_{11} &= c_{11}S_{11}+c_{12}S_{22}+c_{13}S_{33}+2c_{14}S_{23}-e_{11}E_1,\\
T_{22} &= c_{21}S_{11}+c_{22}S_{22}+c_{23}S_{33}+2c_{24}S_{23}-e_{12}E_1,\\
T_{33} &= c_{31}S_{11}+c_{32}S_{22}+c_{33}S_{33}+2c_{34}S_{23}-e_{13}E_1,\\
T_{23} &= c_{41}S_{11}+c_{42}S_{22}+c_{43}S_{33}+2c_{44}S_{23}-e_{14}E_1,\\
T_{31} &= 2c_{55}S_{31}+2c_{56}S_{12}-e_{25}E_2-e_{35}E_3,\\
T_{12} &= 2c_{56}S_{31}+2c_{66}S_{12}-e_{26}E_2-e_{36}E_3,\\
D_1 &= e_{11}S_{11}+e_{12}S_{22}+e_{13}S_{33}+2e_{14}S_{14}+\varepsilon_{11}E_1,\\
D_2 &= 2e_{25}S_{31}+2e_{26}S_{12}+\varepsilon_{22}E_2+\varepsilon_{23}E_3,\\
D_3 &= 2e_{35}S_{31}+2e_{36}S_{12}+\varepsilon_{23}E_2+\varepsilon_{33}E_3.
\end{aligned} \tag{4}$$

In (4), the c_{pq} are the elastic stiffnesses at constant electric field, the ε_{ij} are the dielectric permittivities and the e_{ip} are the piezoelectric stress constants.

We shall consider simple x_1-thickness-shear modes in an infinite plate bounded by planes $x_2 = \pm b$ which separate the plate from a vacuum. In that case, in $|x_2| \leq b$,

$$u_1 = u_1(x_2, t), \qquad u_2 = u_3 = 0; \qquad E_1 = 0, \qquad E_2 = E_2(x_2, t), \qquad E_3 = E_3(x_2, t), \tag{5}$$

and consequently equations (1)–(4) reduce to

$$T_{12,2} = \rho\ddot{u}_1, \tag{6}$$

$$E_{3,2} = -\dot{B}_1, \qquad B_{1,2} = -\mu_0\dot{D}_3, \qquad \dot{D}_2 = 0, \tag{7}$$

$$S_{12} = \tfrac{1}{2}u_{1,2}, \tag{8}$$

$$\begin{aligned}
T_{12} &= 2c_{66}S_{12}-e_{26}E_2-e_{36}E_3,\\
D_2 &= 2e_{26}S_{12}+\varepsilon_{22}E_2+\varepsilon_{23}E_3,\\
D_3 &= 2e_{36}S_{12}+\varepsilon_{23}E_2+\varepsilon_{33}E_3.
\end{aligned} \tag{9}$$

The remaining equations are either satisfied identically or are not coupled with (6)–(9). The latter may be condensed, finally, to the three equations

$$\begin{aligned}
c_{66}u_{1,2}-e_{26}E_{2,2}-e_{36}E_{3,2} &= \rho\ddot{u}_1,\\
e_{26}u_{1,2}+\varepsilon_{22}E_2+\varepsilon_{23}E_3 &= 0,\\
e_{36}\ddot{u}_{1,2}+\varepsilon_{23}\ddot{E}_2+\varepsilon_{33}\ddot{E}_3 &= \mu_0^{-1}E_{3,22}.
\end{aligned} \tag{10}$$

Outside the plate, $u_1 = E_2 = 0$, $E_3 \to E_3^0$, $e_{36} = \varepsilon_{23} = 0$ and $\varepsilon_{33} \to \varepsilon_0$, the permittivity of a vacuum. Equations (10) then reduce to

$$E^0_{3,22} = \varepsilon_0\mu_0\ddot{E}^0_3. \tag{11}$$

For the mechanical boundary conditions on $x_2 = \pm b$, we shall specify oscillating tractions $n_i T_{ij}$, where $\mathbf{n}$ is the unit outward normal. The electromagnetic boundary conditions require continuity of

$$n_i D_i,\ n_j \varepsilon_{ijk} E_k,\ n_i B_i,\ n_j \varepsilon_{ijk} B_k$$

across $x_2 = \pm b$. In view of (5), the boundary conditions reduce to

$$T_{12} = T\cos\omega t, \qquad E_3 = E_3^0, \qquad E_{3,2} = E_{3,2}^{(0)} \tag{12}$$

on $x_2 = \pm b$, where T is the amplitude of the prescribed surface traction.

3. SOLUTION

Anticipating, from the forms of equations (10–12), that there will be two wave numbers and two phase components, we take

$$\begin{aligned} u_1 &= \sum_r A_r \sin\eta_r x_2 \cos(\omega t - \varepsilon_r), \qquad r = 1, 2, \\ E_2 &= \sum_r B_r \cos\eta_r x_2 \cos(\omega t - \varepsilon_r), \qquad r = 1,2, \\ E_3 &= \sum_r C_r \cos\eta_r x_2 \cos(\omega t - \varepsilon_r), \qquad r = 1, 2 \end{aligned} \tag{13}$$

for $|x_2| \le b$ and, for $|x_2| \ge b$,

$$E_3^0 = C_0 \cos\eta_0(x_2 \mp b \mp ct \mp \delta), \tag{14}$$

where $c(= \varepsilon_0^{-\frac{1}{2}}\mu_0^{-\frac{1}{2}})$ is the velocity of electromagnetic waves in a vacuum, $\eta_0 = \omega/c$ and the upper and lower signs are for $x_2 \ge b$ and $x_2 \le b$, respectively.

Upon substitution of (13) in (10), there results

$$\begin{aligned} (\rho\omega^2 - \eta_r^2 c_{66})A_r + \eta_r e_{26} B_r + \eta_r e_{26} C_r &= 0, \\ \eta_r e_{26} A_r + \varepsilon_{22} B_r + \varepsilon_{23} C_r &= 0, \\ \eta_r e_{36} A_r + \varepsilon_{23} B_r + (\varepsilon_{33} - \eta_r^2 \mu_0^{-1}\omega^{-2})C_r &= 0. \end{aligned} \tag{15}$$

From these equations follow the amplitude ratios

$$\begin{aligned} \beta_r &= \frac{B_r}{A_r} = \frac{(c_{66} + e_{26}e_{36}/\varepsilon_{23})\eta_r^2 - \rho\omega^2}{\eta_r(e_{26} - e_{36}\varepsilon_{22}/\varepsilon_{23})}, \\ \gamma_r &= \frac{C_r}{A_r} = \frac{\varepsilon_{22}[\rho\omega^2 - (c_{66} + e_{26}^2/\varepsilon_{22})\eta_r^2]}{\eta_r \varepsilon_{23}(e_{26} - e_{36}\varepsilon_{22}/\varepsilon_{23})}, \end{aligned} \tag{16}$$

in which the η_r are the two positive roots of the equation obtained by setting the determinant of the coefficients of A_r, B_r and C_r, in (15), equal to zero:

$$\eta^4 - \alpha_2\eta^2 + \alpha_3 = 0, \tag{17}$$

where

$$\alpha_2 = \left\{1+\frac{\hat{v}^2}{\hat{c}^2}\left[1+\frac{e_{26}^2}{\varepsilon_{22}c_{66}}+\frac{e_{36}^2}{\varepsilon_{33}c_{66}}-\frac{\varepsilon_{23}}{\varepsilon_{22}}\left(\frac{\varepsilon_{23}}{\varepsilon_{22}}+\frac{2e_{26}e_{36}}{\varepsilon_{22}c_{66}}\right)\right]\right\}\frac{\omega^2}{\hat{v}^2},$$
$$\alpha_3 = \left(1-\frac{\varepsilon_{23}^2}{\varepsilon_{22}\varepsilon_{33}}\right)\frac{\omega^4}{\hat{v}^2\hat{c}^2} \tag{18}$$

and

$$\hat{v}^2 = (c_{66}+e_{26}^2/\varepsilon_{22})/\rho = \hat{c}_{66}/\rho, \qquad \hat{c}^2 = c^2\varepsilon_0/\varepsilon_{33}, \tag{19}$$

i.e. $\hat{v}$ is the velocity of the x_1-thickness-shear wave including the electric, but not the magnetic, influence and c is the velocity of an electromagnetic wave, in the quartz plate, with wave normal and electric vector in the x_2 and x_3 directions, respectively.

The functions (13) may now be written as

$$u_1 = \sum_r A_r \sin\eta_r x_2 \cos(\omega t-\varepsilon_r), \qquad r = 1, 2,$$
$$E_2 = \sum_r \beta_r A_r \cos\eta_r x_2 \cos(\omega t-\varepsilon_r), \qquad r = 1, 2, \tag{20}$$
$$E_3 = \sum_r \gamma_r A_r \cos\eta_r x_2 \cos(\omega t-\varepsilon_r), \qquad r = 1, 2,$$

where β_r and γ_r are given by (16) and η_r by the two positive roots of (17).

Substitution of (20) in the three boundary conditions (12), yields the following six equations on the six constants A_r', A_r'', C_0', C_0'':

$$a_1A_1'+a_2A_2' = T, \qquad a_1A_1''+a_2A_2'' = 0,$$
$$b_1A_1'+b_2A_2'-C_0' = 0, \qquad b_2A_1''+b_2A_2''+C_0'' = 0, \tag{21}$$
$$c_1A_1'+c_2A_2'+\eta_0C_0'' = 0, \qquad c_1A_1''+c_2A_2''+\eta_0C_0' = 0,$$

where

$$A_r' = A_r\cos\varepsilon_r, \qquad A_r'' = A_r\sin\varepsilon_r, \qquad r = 1, 2,$$
$$C_0' = C_0\cos\delta, \qquad C_0' = C_0\sin\delta, \tag{22}$$

$$a_r = (\eta_r c_{66}-\beta_r e_{26}-\gamma_r e_{36})\cos\eta_r b, \qquad r = 1, 2,$$
$$b_r = \gamma_r\cos\eta_r b, \qquad c_r = \eta_r\gamma_r\sin\eta_r b, \qquad r = 1, 2. \tag{23}$$

The solution of the six equations (21) is

$$A_1' = (b_2\eta_0^2\Delta_{ab}-c_2\Delta_{ca})T/\Delta, \qquad A_1'' = a_2\eta_0\Delta_{bc}T/\Delta,$$
$$A_2' = -(b_1\eta_0^2\Delta_{ab}-c_1\Delta_{ca})T/\Delta, \qquad A_2'' = -a_1\eta_0\Delta_{bc}T/\Delta, \tag{24}$$
$$C_0' = -\Delta_{bc}\Delta_{ca}T/\Delta, \qquad C_0'' = \eta_0\Delta_{ab}\Delta_{bc}T/\Delta,$$

where

$$\Delta_{ab} = a_1b_2-a_2b_1, \qquad \Delta_{bc} = b_1c_2-b_2c_1,$$
$$\Delta_{ca} = c_1a_2-c_2a_1, \qquad \Delta = \eta_0^2\Delta_{ab}^2+\Delta_{ca}^2. \tag{25}$$

This completes the formal solution of the problem.

4. POWER RADIATED

Since the only power loss from the vibrating plate is through electromagnetic radiation, the radiated power must be equal to the power input of the surface traction. Per unit area of each face of the plate, this is $T_{12}\dot{u}_1$, on $x_2 = \pm b$. The radiated power, per unit area of each face, averaged over the period $\tau(=2\pi/\omega)$, is

$$P = \tau^{-1}\int_0^\tau (T_{11}\dot{u}_1)_{x_2=b}\,dt = \tfrac{1}{2}\omega T\sum_r A_r'' \sin\eta_r b. \tag{26}$$

A simple formula for P may be found after taking into account the magnitudes of the terms involved in η_r and A_r''.

For the AT-cut of quartz [3], the constants c, e and ε, appearing in (16), as calculated by Syke's formulae [4] from Bechmann's [5] values of the principal constants, are

$$\begin{aligned}
c_{66} &= 29{\cdot}013\times10^{10}\ \text{dyn/cm}^2\\
e_{26} &= -9{\cdot}490\times10^{-6}\ \text{C/cm}^2\\
e_{36} &= 6{\cdot}707\times10^{-6}\ \text{C/cm}^2\\
\varepsilon_{22} &= 39{\cdot}816\times10^{-21}\ \text{C}^2/\text{dyn cm}^2\\
\varepsilon_{33} &= 40{\cdot}424\times10^{-21}\ \text{C}^2/\text{dyn cm}^2\\
\varepsilon_{23} &= 0{\cdot}8678\times10^{-21}\ \text{C}^2/\text{dyn cm}^2.
\end{aligned}$$

Also,

$$\begin{aligned}
\rho &= 2{\cdot}6485\ \text{g/cm}^3\\
\hat{v} &= 3{\cdot}322\times10^{5}\ \text{cm/sec}\\
\varepsilon_0 &= 8{\cdot}854\times10^{-21}\ \text{C}^2/\text{dyn cm}^2\\
c &= 2{\cdot}998\times10^{10}\ \text{cm/sec}.
\end{aligned}$$

Accordingly, to a close approximation,

$$\alpha_2 = \omega^2/\hat{v}^2, \qquad \alpha_3 = \omega^4/\hat{v}^2\hat{c}^2 \tag{27}$$

and

$$\eta_1^2 = (\omega^2/\hat{v}^2)(1-\hat{v}^2/\hat{c}^2), \qquad \eta_2^2 = \omega^2/\hat{c}^2. \tag{28}$$

Thus, to the second degree in $\hat{v}/\hat{c}$, i.e. to 10^{-10}, the wave number of the essentially mechanical wave is slightly less than if the electromagnetic effect is neglected and the wave number of the electromagnetic wave is not influenced by the elasticity. From (28), it is apparent that the percent increase in resonance frequencies due to the electromagnetic effect is only about 6×10^{-7}.

We shall take the forcing frequency, ω, to be a resonance frequency as calculated in the absence of the electromagnetic field:

$$\omega = n\pi\hat{v}/2b, \qquad n \text{ odd}. \tag{29}$$

Then,

$$\eta_0 = n\pi\hat{v}/2bc, \qquad \eta_1 \approx n\pi(1-\tfrac{1}{2}\hat{v}^2/\hat{c}^2)/2b, \qquad \eta_2 \approx n\pi\hat{v}/2b\hat{c} \tag{30}$$

and

$$\cos\eta_1 b \approx -(-1)^{2n-1} n\pi\hat{v}^2/4\hat{c}^2, \qquad \cos\eta_2 b \approx 1,$$
$$\sin\eta_1 b \approx -(1)^{2n-1}, \qquad \sin\eta_2 b \approx n\pi\hat{v}/2\hat{c}. \tag{31}$$

The amplitude of the strain in the middle plane of the plate is

$$S_0 = |u_{1,2}|_{x_2=0} \approx A_1''\eta_1 \approx a_2\eta_0\eta_1\Delta_{bc}T/\Delta \tag{32}$$

and P is, approximately,

$$P \approx \tfrac{1}{2}\omega A_1''T \approx \tfrac{1}{2}\omega\Delta S_0^2/a_2\eta_0\eta_1^2\Delta_{bc}. \tag{33}$$

Now,

$$\Delta \approx \Delta_{ca}^2 \approx \rho^4\omega^8(\hat{v}\varepsilon_{22}/\hat{c}\varepsilon_{23})^2(2b/n\pi)^2/e_{26}^2(1-e_{36}\varepsilon_{22}/e_{26}\varepsilon_{23})^2.$$
$$\Delta_{bc} \approx \rho^2\omega^4(\hat{v}/\hat{c})(\varepsilon_{22}/\varepsilon_{23})^2(2b/n\pi)/e_{26}^2(1-e_{36}\varepsilon_{22}/e_{26}\varepsilon_{23})^2,$$
$$a_2 \approx \rho\omega^2/\eta_2, \qquad \rho\omega^2 \approx \hat{c}_{66}(n\pi/2b)^2.$$

Consequently we have, finally,

$$P \approx \tfrac{1}{2}\varkappa_{33}\hat{c}_{66}^2 S_0^2/\rho c, \tag{34}$$

where $\varkappa_{33} = \varepsilon_{33}/\varepsilon_0$, as the simple formula for the power radiated, per unit area, from each face of the plate. Employing the numerical values of the constants, we have

$$P \approx 25\times 10^4\, S_0^2\ \mathrm{W/cm^2}. \tag{35}$$

Hence, for a strain of 10^{-5}, the power radiated is about 25 μW/cm^2.

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Абстракт—Решаются линейные уравнения пьезоэлектромагнетизма для случая сдвиговых колебаний по толщине для пластинки из кварца. Для пластинки среза *AT*, колебающейся близи резонанса, излучение из каждой стороны является приблизительно 25 мквт/см² для деформации сдвига 10^{-5}.